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A CHARACTERIZATION OF RIEMANN'S MINIMAL SURFACES

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Abstract

We prove that Riemann's minimal surfaces are the only properly embedded minimal tori with two planar ends in $\mathbb{R}^3/T$, where T is the group generated by a nontrivial translation in $\mathbb{R}^3$. In the proof of this result we find all the properly immersed minimal tori with two parallel embedded planar ends. The space of such surfaces is described by regular curves, parameterized by $\mathbb{R}$, in the moduli space of conformal structures on a topological torus. Except in the case of Riemann's minimal surfaces, these curves contain points which yield minimal surfaces with vertical flux, and hence the surfaces are not embedded.

Introduction

In a paper published in 1867, Riemann [21] found a one-parameter family of complete, embedded, singly-periodic minimal surfaces foliated by circles and lines in parallel planes. These surfaces, known nowadays as Riemann examples or Riemann's minimal surfaces, were characterized by Riemann in [21] as the only minimal surfaces fibered by circles in parallel planes besides the catenoid. Since then many different characterization results have been proved for the surfaces.

Enneper [6] proved around 1870 that a minimal surface fibered by circular arcs was in fact a piece of a Riemann example or a piece of the catenoid. Shiffman [23] proved in 1956 that a minimal annulus spanning

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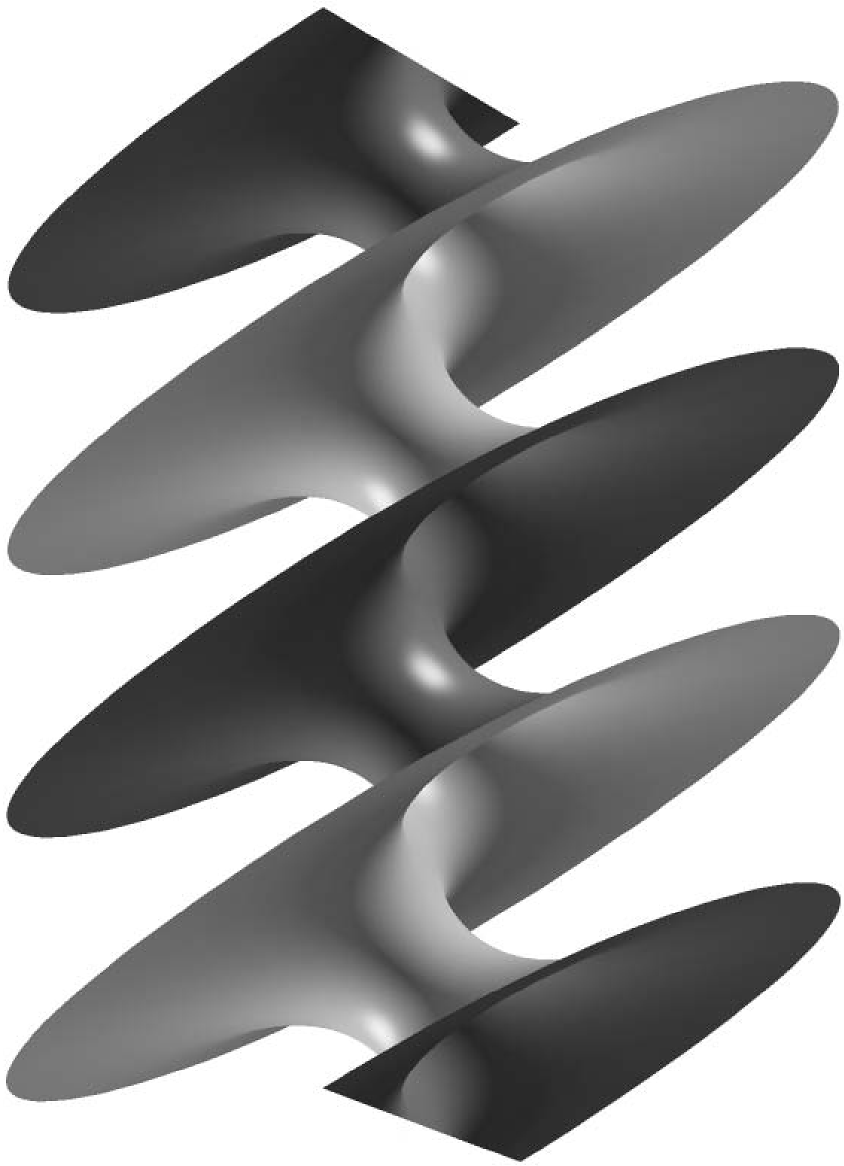

FIGURE 1. A Riemann example.

two circles in parallel planes was foliated by circles in parallel planes and hence a piece of a Riemann example or a piece of the catenoid. See also [10] for a good survey.

In 1991 Hoffman, Karcher and Rosenberg [9] proved that an embedded minimal annulus with boundary of two parallel lines on parallel planes and lying between the planes extends by Schwarz reflection to a Riemann example. Toubiana [24] proved in 1992 that there are no minimal annuli bounded by two nonparallel lines in parallel planes. A more general result was proved by Pérez and Ros [20] in 1993; they showed that there are no properly embedded minimal surfaces of genus one with a finite number of planar ends in $\mathbb{R}^3/S_\theta$, $\theta \neq 0$. In 1993 Romon [22] proved that a properly embedded annulus with one flat end, lying between two parallel planes and bounded by two parallel lines in the planes, is a piece of a Riemann example. We should also mention that Pérez [19] has recently proved that a properly embedded minimal torus in $\mathbb{R}^3/T$ with two planar type ends is a Riemann example provided it is symmetric with respect to a plane. A. Douady and R. Douady [5] have proved that Riemann examples are the only singly-periodic (with translational symmetries) minimal surfaces of genus one with planar ends and a symmetry with respect to a plane. Fang [7] showed that a properly embedded minimal annulus in a slab with boundary consisting of two circles or lines (all combinations allowed) must be part of a Riemann example. Also Fang and Wei [26] have proved that an annulus with a planar end and boundary consisting of circles or lines in parallel planes is a piece of a Riemann example.

In this paper we give another characterization of the Riemann examples. We prove in Theorem 3.1 that

> *The only embedded minimal tori with two planar ends in* $\mathbb{R}^3/T$ *are Riemann's minimal surfaces.*

We have denoted by $\mathbb{R}^3/T$ the quotient of $\mathbb{R}^3$ by the group T generated by a nontrivial translation. The proof is based on finding *all* the complete immersed minimal tori with two parallel embedded planar ends (Figure 2 and Figure 3). We prove in Theorem 3.2 that

> *The space of singly-periodic minimal immersed tori in* $\mathbb{R}^3$ *with two parallel embedded planar ends, viewed on the moduli space of genus-one conformal structures, consists of a countable number of regular curves.*

Moreover, except in the case of Riemann examples, each one of these curves contains at least one point which gives a surface with vertical flux.

The above curves are parameterized by certain homology classes on a topological genus-one surface.

Our main result stated in Theorem 3.1 implies the ones of [9] and [19].

The technique used in this work consists of an analysis of the periods as holomorphic functions of the conformal structures. It was first used by López [13] to characterize Chen-Gackstatter surface as the only complete minimal torus with total curvature -8π. Minimal surfaces with vertical flux admits a deformation by minimal surfaces preserving the conformal structure. By using this deformation it can be proved that such a surface has transversal self-intersections; see the papers by López and Ros [14] and Pérez and Ros [20].

We conclude this section by providing with some further background in the study of embedded minimal surfaces that are related to our work. In $\mathbb{R}^3$, Riemann examples are surfaces of genus zero with infinite number of planar ends. It is conjectured by Meeks [15] and Rosenberg [17] that Riemann examples, together with the plane, the catenoid and the helicoid, are the only properly embedded minimal surfaces of genus zero in $\mathbb{R}^3$. A weaker conjecture is that the Riemann examples are the only properly embedded minimal surface of genus zero with infinite number of ends and infinite symmetry group in $\mathbb{R}^3$. By a theorem of Callahan, Hoffman and Meeks [2] and the result of Perez and Ros mentioned earlier, the latter conjecture is equivalent to saying that Riemann examples are the only properly embedded minimal surfaces of genus-one with even number of ends in the quotient space $\mathbb{R}^3/T$. Our result shows that it is true when the number of ends is two, which is the minimum. It also should be pointed out the genus-one assumption is crucial, as there are examples of higher genus embedded minimal surface in $\mathbb{R}^3/T$: Callahan, Hoffman, Meeks [1] constructed a family of embedded minimal surfaces of odd genus (greater than one) and two planar ends in $\mathbb{R}^3/T$. Using numerical methods, the third author constructed a proprely embedded minimal surface of genus two and two planar ends in $\mathbb{R}^3/T$ by adding handles to the Riemann examples [25]. Finally, there are also embedded minimal surfaces in $\mathbb{R}^3/T$ that have helicoid type ends and Scherk type ends; interested reader can refer [12].

The paper is organized as follows. In the next section we describe

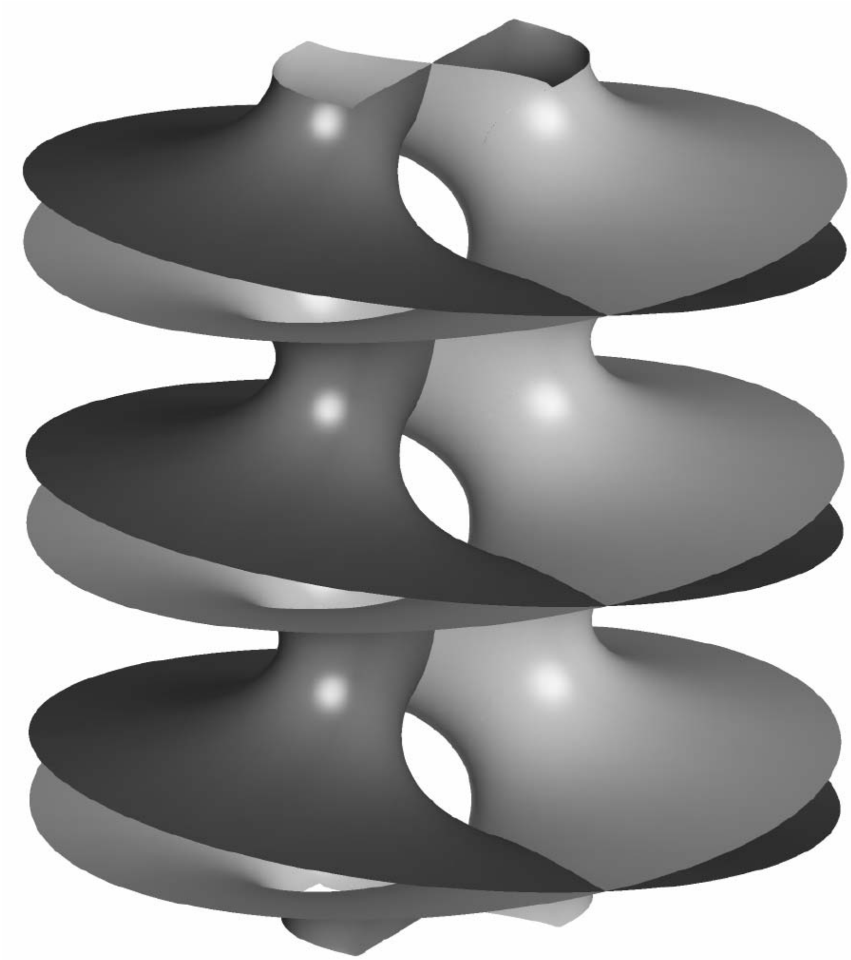

Figure 2. An immersed example of non-vertical flux. The image shown here consists of three pieces of the fundmental domain, each has two planar ends intersecting each other and asymptotic to the same plane at infinity.

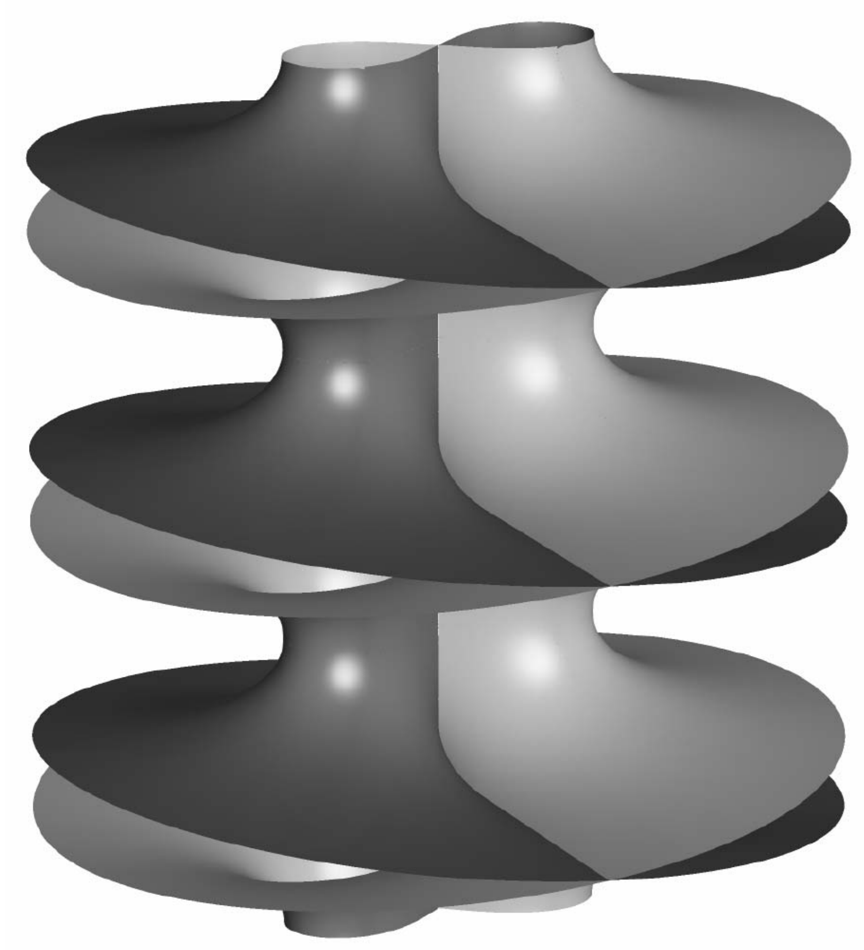

Figure 3. An immersed example of vertical flux.

the Weierstrass representation for singly-periodic minimal surfaces. In the second section we reduce the period problem to the study of the nodal set of a harmonic function. In the last section we state and prove our main results.

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1. Preliminaries

In this work we shall use the Weierstrass representation for singly-periodic minimal surfaces; see [16]. Let $\psi : M \to \mathbb{R}^3/T$ be a conformal minimal immersion from a Riemann surface into a quotient of $\mathbb{R}^3$ by the group T generated by a nontrivial translation. Let $\phi : M \to \mathbb{S}^2$ be the Gauss map and $g : M \to \overline{\mathbb{C}}$ the stereographic projection of the Gauss map. Then we have

$$d'\psi = (1 - g^2, i\,(1 + g^2), 2g)\,\omega,$$

where d' is the holomorphic part of the differential on the Riemann surface, and ω is a holomorphic one-form on M; see [18] and [11]. Conversely, if g and ω are a meromorphic map and a holomorphic one-form on the Riemann surface M, then

$$\widetilde{\psi} = \mathrm{Re}\int (1 - g^2, i\,(1 + g^2), 2g)\,\omega \tag{1}$$

defines a multivalued branched conformal minimal immersion $\widetilde{\psi} : M \to \mathbb{R}^3$. The immersion $\widetilde{\psi}$ is unbranched if the zeroes of ω coincide with the poles of g and have double order. $\widetilde{\psi}$ induces a minimal immersion $\psi : M \to \mathbb{R}^3/T$ if and only if the group of periods

$$\{\mathrm{Per}_\alpha(\widetilde{\psi});\ \alpha \in H_1(M, \mathbb{Z})\}$$

coincides with T, where $\mathrm{Per}_\alpha \widetilde{\psi}$ is the real part of the integral of (1) along α.

Suppose now that $M \subset \mathbb{R}^3/T$ is an immersed minimal torus with two properly embedded parallel planar ends. The surface M is conformally equivalent to a torus $\overline{M}$ with two points removed. The Gauss map $g : M \to \overline{\mathbb{C}}$ extends meromorphically to $\overline{M}$, and the total curvature of M is given by

$$C(M) = \int_M K = 2\pi\,(\chi(M) - 2) = -8\pi.$$

Since $-4\pi \deg\ g = C(M)$, the degree of the Gauss map is 2; see [16] and [18].

Without loss of generalality, we can assume that normal vectors at the ends are vertical. Since the ends are of planar type, we know that the Gauss map is branched at the ends (see [16]). We can assume further that $\overline{M}$ is conformally diffeomorphic to the Riemann surface $\overline{M}_a$ of the algebraic curve $w^2 = (z^2-1)(z-a)$, with $a \in \mathbb{C} - \{+1, -1\}$, that M is given by $\overline{M_a} - \{(a,0), (\infty,\infty)\}$, and that the Gauss map g is given, after a rotation around the x_3 axis if necessary, by $g(z,w) = r\,(z-a)$, with $r > 0$. The one-form ω then is given, up to scaling, by $\omega(z,w) = \theta\,dz/(r\,(z-a)\,w)$, with $\theta \in \mathbb{C}$, $|\theta| = 1$.

Summarizing the previous discussion we have

Proposition 1.1. *Let $\psi : M \to \mathbb{R}^3/T$ be a minimal immersion of a torus with two parallel embedded planar ends. Then M is conformally equivalent to*

$$\overline{M}_a - \{(a,0), (\infty,\infty)\}, \tag{2}$$

for some $a \in \mathbb{C} - \{\pm 1\}$, with Weierstrass data

$$g = r\,(z-a), \quad \omega = \frac{\theta}{r\,(z-a)}\frac{dz}{w}, \quad r > 0,\ |\theta| = 1. \tag{3}$$

We are now going to deal with the problem of periods in the next proposition:

Proposition 1.2. *Let M be given by* (2)*. Consider the meromorphic map and the holomorphic one-form on $\overline{M}_a$ given by* (3)*. Let $\{\alpha, \beta\}$ be a homology basis on $\overline{M}_a$ and*

$$f_\alpha(a) = \int_\alpha (z-a)\frac{dz}{w}, \qquad h_\alpha(a) = \int_\alpha \frac{dz}{w}. \tag{4}$$

Then we have the following:

(i) *If $f_\alpha(a) = 0$, then, up to sign, there exists a unique θ such that* (3) *induce a minimal immersion $\psi_r : M \to \mathbb{R}^3/T_r$ with $\mathrm{Per}_\alpha(\psi_r) = 0$ for each $r > 0$, where $T_r = \langle \mathrm{Per}_\beta(\psi_r)\rangle$.*

(ii) If $f_\alpha(a) \neq 0$, but

$$\mathrm{Re}\left(\frac{1}{(a^2-1)^{1/2}}\frac{f_\alpha(a)}{h_\alpha(a)}\right) = 0, \quad r^2 = |a^2-1|^{-1}, \tag{5}$$

then, up to sign, there exists a unique θ such that (3) *induce a minimal immersion $\psi : M \to \mathbb{R}^3/T$ with $\mathrm{Per}_\alpha(\psi) = 0$ and $T = \langle \mathrm{Per}_\beta(\psi)\rangle$.*

Moreover, if $\psi : M \to \mathbb{R}^3/T$ is a minimal immersion with Weierstrass data given by (3) *on the Riemann surface $w^2 = (z^2-1)(z-a)$ minus the points $(a,0)$, (∞,∞) and $\mathrm{Per}_\alpha(\psi) = 0$, then either $f_\alpha(a) = 0$ or $f_\alpha(a) \neq 0$ and* (5) *holds.*

Proof. The Weierstrass data in (3) induce an unbranched multivalued conformal minimal immersion with two planar parallel embedded ends. We only have to consider the problem of periods. First observe that $\mathrm{Per}_\gamma(\psi)$ vanishes for a single closed loop γ around any of the ends. So the only periods arise from cycles in $H_1(\overline{M_a}, \mathbb{Z})$. The Weierstrass data will produce a minimal surface in $\mathbb{R}^3/T$, where T is the group generated by a translation if and only is there exists a homology basis $\{\alpha, \beta\}$ such that $\mathrm{Per}_\alpha(\psi) = 0$. Note that in this case $\mathrm{Per}_\beta(\psi) \neq 0$ since otherwise we would have a minimal surface in $\mathbb{R}^3$ with third coordinate ψ_3 bounded because of the planar type of the ends. But the maximum principle for harmonic functions on $\overline{M}_a$ would imply that ψ_3 is constant, giving us a contradiction.

To prove (i) and (ii), denote $(\phi_1, \phi_2, \phi_3) = (1-g^2, i\,(1+g^2), 2g)\,\omega$. Observe that

$$2\,d\left(\frac{z^2-1}{w}\right) = (z-a)\,\frac{dz}{w} + (1-a^2)\,\frac{1}{z-a}\,\frac{dz}{w}, \tag{6}$$

and so $dz/((z-a)\,w)$ and $(z-a)\,dz/((a^2-1)\,w)$ differ in an exact one-form. We have

$$\begin{aligned}\int_\alpha \phi_1 &= \int_\alpha \theta\left(\frac{1}{r\,(z-a)} - r\,(z-a)\right)\frac{dz}{w} \\ &= \theta\left(\frac{1}{r\,(a^2-1)} - r\right)\, f_\alpha(a),\end{aligned}$$

$$\int_\alpha \phi_2 = \int_\alpha i\theta\left(\frac{1}{r\,(z-a)} + r\,(z-a)\right)\frac{dz}{w}$$
$$= i\theta\left(\frac{1}{r\,(a^2-1)} + r\right) f_\alpha(a), \tag{7}$$
$$\int_\alpha \phi_3 = \int_\alpha 2\theta\frac{dz}{w} = 2\theta h_\alpha(a).$$

If $f_\alpha(a) = 0$ then we have that $\int_\alpha \phi_1 = \int_\alpha \phi_2 = 0$. We can choose a unique θ, up to sign, so that $\operatorname{Re}\int_\alpha \phi_3 = \operatorname{Re}(2\theta h_\alpha(a)) = 0$. From the above observations we obtain that, for every $r > 0$, we have Weierstrass data given by (3) that induce minimal immersions $\psi_r : M \to \mathbb{R}^3/T_r$, where T_r is a nontrivial group generated by a translation. This proves (ii).

Suppose now that (5) holds. As above we can choose a unique θ, up to sign, such that $\operatorname{Re}(2\theta h_\alpha(a)) = 0$. Let us see now that $\operatorname{Re}\int_\alpha \phi_1 = \operatorname{Re}\int_\alpha \phi_2 = 0$.

Recall that $f_\alpha(a)/((a^2-1)^{1/2}h_\alpha(a))$, $\theta h_\alpha(a) \in i\mathbb{R}$. From this and the formulae

$$\theta\left(\frac{1}{a^2-1} - r^2\right) f_\alpha(a)$$
$$= \theta h_\alpha(a)\left(\frac{1}{a^2-1} - r^2\right)(a^2-1)^{1/2}\frac{1}{(a^2-1)^{1/2}}\frac{f_\alpha(a)}{h_\alpha(a)},$$
$$i\theta\left(\frac{1}{a^2-1} + r^2\right) f_\alpha(a)$$
$$= i\theta h_\alpha(a)\left(\frac{1}{a^2-1} + r^2\right)(a^2-1)^{1/2}\frac{1}{(a^2-1)^{1/2}}\frac{f_\alpha(a)}{h_\alpha(a)},$$

we have that $\operatorname{Re}\int_\alpha \phi_1 = \operatorname{Re}\int_\alpha \phi_2 = 0$ if and only if

$$((a^2-1)^{-1} - r^2)(a^2-1)^{1/2} \text{ and } i((a^2-1)^{-1} + r^2)(a^2-1)^{1/2}$$

are in $i\mathbb{R}$ and this is equivalent to $r^2 = |a^2-1|^{-1}$. So (ii) holds.

Finally, suppose that $\psi : M \to \mathbb{R}^3/T$ is a minimal immersion with $\operatorname{Per}_\alpha(\psi) = 0$. Then squaring the first two lines in (7), adding them and extracting square root we obtain that $f_\alpha(a)/((a^2-1)^{1/2}h_\alpha(a)) \in i\mathbb{R}$. If $f_\alpha(a) \neq 0$, then dividing the first line in (7) by the second one we have that $((a^2-1)^{-1} - r^2)/(i\,(a^2-1)^{-1} + r^2)) \in \mathbb{R}$, and this is equivalent to $r^2 = |a^2-1|^{-1}$. q.e.d.

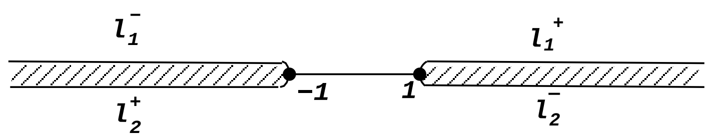

FIGURE 4. Boundary of Ω.

2. Analysis of the period problem

We consider the algebraic curve $\{(a, y) \in \overline{\mathbb{C}}^2;\ y^2 = a^2 - 1\}$, and denote by Ω the region in $a^{-1}(\overline{\mathbb{C}} - ([\infty, -1] \cup [1, \infty]))$ which contains the point $(0, i)$. Identify the points in Ω with the complex numbers in $a(\Omega)$. Let $\overline{\Omega}$ be the closure of Ω on the Riemann surface of the polynomial $y^2 = a^2 - 1$. The boundary of Ω is given by $\ell_1 = \ell_1^+ \cup \ell_1^-$ and $\ell_2 = \ell_2^+ \cup \ell_2^-$, where

$$\ell_1^- = \{a^{-1}([-\infty, -1]);\ y < 0\}, \qquad \ell_1^+ = \{a^{-1}([1, +\infty]);\ y > 0\},$$
$$\ell_2^+ = \{a^{-1}([-\infty, -1]);\ y > 0\}, \qquad \ell_2^- = \{a^{-1}([1, +\infty]);\ y < 0\};$$

see Figure 4. We shall also denote by ∞_i the preimages of the point $\infty \in \overline{\mathbb{C}}$ on the lines ℓ_i, $i = 1, 2$.

Define for $a \in \Omega$ the holomorphic functions:

$$f_1(a) = \int_{[-1,a]} \frac{x - a}{\sqrt{(x^2 - 1)(x - a)}}\, dx,$$
$$f_2(a) = \int_{[a,1]} \frac{x - a}{\sqrt{(x^2 - 1)(x - a)}}\, dx,$$
$$h_1(a) = \int_{[-1,a]} \frac{1}{\sqrt{(x^2 - 1)(x - a)}}\, dx,$$
$$h_2(a) = \int_{[a,1]} \frac{1}{\sqrt{(x^2 - 1)(x - a)}}\, dx,$$

where we have chosen the single valued branch of $\sqrt{(x^2 - 1)(x - a)}$ on $\bigcup_{a \in \Omega}[-1, a]$ for f_1 and h_1 and on $\bigcup_{a \in \Omega}[a, 1]$ for f_2 and h_2 determined by $h_1(0) > 0$ ($\Rightarrow f_1(0) < 0$), $ih_2(0) > 0$ ($\Rightarrow if_2(0) > 0$); see [13, proof of Lemma 1] for complete details.

The functions f_i, h_i are half the integrals of the one-forms $(z - a)\,dz/w$ and dz/w on the Riemann surface $w^2 = (z^2-1)(z-a)$ along the curves α_1, α_2 defined as the closed curves given by the liftings to $\overline{M}_a$ of the slits $[-1,a]$, $[a,1]$, respectively, in the z-plane with the orientation induced by the choice of w given above. Recalling (4) we have $2f_i(a) = f_{\alpha_i}(a)$ and $2h_i(a) = h_{\alpha_i}(a)$.

Straightforward analytic continuation arguments, see [13, Remark 4], show that the distribution of the values of f_1, h_1, f_2, h_2 on $\ell_1 \cup \ell_2$ are

(8)

	f_1	f_2	h_1	h_2
ℓ_1^+	$\mathbb{R}_- + i\mathbb{R}_-$	$i\mathbb{R}_+$	$\mathbb{R}_+ + i\mathbb{R}_+$	$i\mathbb{R}_-$
ℓ_2^-	$\mathbb{R}_- + i\mathbb{R}_+$	$i\mathbb{R}_+$	$\mathbb{R}_+ + i\mathbb{R}_-$	$i\mathbb{R}_-$
ℓ_1^-	$\mathbb{R}_+$	$\mathbb{R}_- + i\mathbb{R}_-$	$\mathbb{R}_+$	$\mathbb{R}_- + i\mathbb{R}_-$
ℓ_2^+	$\mathbb{R}_+$	$\mathbb{R}_+ + i\mathbb{R}_-$	$\mathbb{R}_+$	$\mathbb{R}_+ + i\mathbb{R}_-$
$(-1,1)$	$\mathbb{R}_-$	$i\mathbb{R}_-$	$\mathbb{R}_+$	$i\mathbb{R}_-$

Where $\mathbb{R}_+ = (0,+\infty)$ and $\mathbb{R}_- = (-\infty,0)$. From this we obtain

$$
\begin{array}{llllll}
i\,\mathrm{Im}(f_1) & = -f_2, & i\,\mathrm{Im}(h_1) & = -h_2, & \text{on} & \ell_1^+ \\
i\,\mathrm{Im}(f_1) & = f_2, & i\,\mathrm{Im}(h_1) & = h_2 & \text{on} & \ell_2^-, \\
\mathrm{Re}(f_2) & = -f_1, & \mathrm{Re}(h_2) & = -h_1 & \text{on} & \ell_1^-, \\
\mathrm{Re}(f_2) & = f_1, & \mathrm{Re}(h_2) & = h_1 & \text{on} & \ell_2^+.
\end{array} \tag{9}
$$

We have the following properties for f_i, h_i, $i = 1,2$.

Lemma 2.1. *The functions f_i, h_i, $i = 1,2$, are holomorphic on Ω and continuous on $\overline{\Omega}$ (taking eventually infinite values on the set $\{\pm 1, \infty_1, \infty_2\}$). Furthermore, the functions f_i, h_i are holomorphic at $(-1)^i$, non holomorphic at $(-1)^{i+1}$ and satisfy*

$$
\begin{aligned}
f_2(a) &= i f_1(-a), \qquad h_2(a) = -i h_1(-a), \\
f_1(\overline{a}) &= \overline{f_1(a)}, \qquad h_1(\overline{a}) = \overline{h_1(a)}.
\end{aligned} \tag{10}
$$

Moreover, calling $f = f_i$, $h = h_i$, we have

$$
f'(a) = -\frac{h(a)}{2}, \quad h'(a) = \frac{f(a)}{2(a^2-1)}, \tag{11}
$$

and that

$$
\begin{aligned}
f''(a) &= \frac{1}{4\,(1-a^2)}\, f(a), \\
h''(a) &= \frac{2a}{1-a^2}\, h'(a) + \frac{1}{4\,(1-a^2)}\, h(a).
\end{aligned} \tag{12}
$$

Proof. The first part of the lemma is elementary and follows from López's work [13]. Equations (10) follow from the definition of the functions f_i and h_i, $i = 1, 2$.

To prove (11) note that for any closed curve $\gamma = n\alpha_1 + m\alpha_2$, n, $m \in \mathbb{Z}$, and any meromorphic 1–form τ on $\overline{M}_a$ we have $d/da(\int_\gamma \tau) = \int_\gamma (\partial\tau/\partial a)$. We apply this observation to the curves α_1, α_2 and to the meromorphic 1-forms dz/w and $(z-a)dz/w$. Hence integrating by parts and using (6) we obtain equation (11).

Equations (12) are deduced from (11). q.e.d.

Define

$$u(a) = \frac{1}{y(a)} \left(\frac{nf_1 + mf_2}{nh_1 + mh_2} \right)(a),$$

and recall that $y(a)$ is the branch of $(a^2 - 1)^{1/2}$ on Ω given at the beginning of this section.

Proposition 2.2. *The function u is holomorphic on Ω and extends continuously (taking eventually the value ∞ on $\{\pm 1\}$) to $\overline{\Omega}$.*

The function $\mathrm{Re}(u)$ is harmonic on Ω and extends continuously to $\overline{\Omega} - \{\pm 1\}$. Their nodal curves stay away from the boundary of Ω except at the points $a = \pm 1$, which only one nodal curve approaches.

Proof. Since h_1, h_2 are the integrals over the homology basis α_1, α_2 of the holomorphic one-form dz/w, note that $nh_1 + mh_2$ never vanishes on $\overline{\Omega} - \{\pm 1, \infty_1, \infty_2\}$; see [8].

By Lemma 2.1 the functions f_i, h_i extends continuously to the boundary of Ω minus the points ± 1 and ∞_i, and so $u(a)$ also extends. It remains to study the points ± 1 and ∞_i.

Let us see first that the nodal curves do not meet the boundary of Ω minus $\{\pm 1, \infty_1, \infty_2\}$. Since m, n are arbitrary, the symmetries in (10) allows us to restrict our attention to the case $a \in \ell_1^+ - \{1, \infty_1\}$.

Taking into account (9) we have

$$
\begin{aligned}
\mathrm{Re}(u(a)) = 0 &\iff \mathrm{Re}\left[\frac{n\ \mathrm{Re}(f_1) + (m-n) f_2}{n\ \mathrm{Re}(h_1) + (m-n) h_2} \right](a) = 0 \\
&\iff \left[n^2\, \mathrm{Re}(f_1)\, \mathrm{Re}(h_1) - (m-n)^2 f_2 h_2 \right](a) = 0,
\end{aligned}
$$

for $a \in \ell_1^+$.

As $\mathrm{Re}(f_1)\,\mathrm{Re}(h_1)(a) < 0$ and $f_2 h_2(a) > 0$, see (8), we deduce that $\mathrm{Re}(u(a)) \neq 0$ for $a \in \ell_1^+ - \{1, \infty_1\}$.

Let us see now that only one nodal curve of $\mathrm{Re}(u)$ approaches the points ± 1. Again by the symmetries in (10) we only need to consider the case $a = 1$.

We know from Lemma 2.1 that f_i and h_i satisfy the second order differential equations (12) with singularities of the first kind at $a = 1$; see [4, Chapter 4]. Hence Frobenius method can be applied to describe the asymptotic behavior of the solutions of these equations around the singularities. The fundamental solutions for the equations that satisfy f_i and h_i, respectively, are

$$\text{for } f_i : \begin{cases} (a-1)\,t_1(a), \\ t_2(a) + t_3(a)(a-1)\log(a-1), \end{cases} \qquad (13)$$
$$\text{for } h_i : \begin{cases} s_1(a), \\ (a-1)\,s_2(a) + s_3(a)\log(a-1), \end{cases}$$

where t_i, s_i are holomorphic functions around 1 and $t_1(1)$, $t_2(1)$, $s_1(1)$, $s_3(1) \neq 0$. By [4, Chapter 4] the Taylor series of the functions t_i, s_i can be taken with real coefficients, i.e., these functions send real numbers to real numbers.

Since f_2, h_2 are holomorphic at 1 and send the points in $(-1,1)$ to imaginary numbers we have

$$f_2(a) = [(a-1)\,t_1(a)]\,i\lambda, \quad h_2(a) = s_1(a)\,i\mu, \quad \lambda, \mu \in \mathbb{R}^*.$$

Write

$$\begin{aligned} f_1(a) = &\rho_1[(a-1)t_1(a)] + \rho_2[t_2(a) + i\pi t_3(a)(a-1)] \\ &+ \rho_2[t_3(a)(a-1)\log(1-a)], \end{aligned} \qquad (14)$$

For $a < 1$, take the imaginary part on (14):

$$\mathrm{Im}(f_1)(a) = \mathrm{Im}(\rho_2)[t_3(a)(a-1)\log(1-a)] + v(a),$$

where $v(a)$ is a harmonic function in a neighborhood of $a = 1$. Since $f_1((-1,1)) \subset \mathbb{R}$ we have $\mathrm{Im}(\rho_2) = 0$. Taking into account (14) and $f_1((-1,1)) \subset \mathbb{R}$ again we deduce that $t_1 = (\text{real constant})t_3$ and it is not hard to conclude that

$$f_1(a) = t_2'(a) + t_3'(a)(a-1)\log(1-a),$$

where $t_2'(a)$, $t_3'(1) \neq 0$, send real numbers to real numbers and $\log(1-a) \in \mathbb{R}$ for $a \in (-\infty, 1)$.

Analogously we have $s_1 = (\text{real constant})s_3$ and

$$h_1(a) = s_2'(a) + s_3'(a)\log(1-a),$$

where $s_3'(1) \neq 0$, s_i' send real numbers to real numbers and $\log(1-a) \in \mathbb{R}$ for $a \in (-\infty, 1)$.

Let us see that the argument of u around small circles centered at $a = 1$ is strictly increasing when $n = 0$ and strictly decreasing when $n \neq 0$. Moreover, the absolute value of the total variation of argument of u goes to π when the radii of the circles tend to zero. Write $a - 1 = re^{i\theta}$, where $r > 0$ and θ is the principal argument of $a - 1$. It is well known that

$$\frac{\partial}{\partial\theta}\arg(u(a)) = \operatorname{Re}\left[(a-1)\frac{u_a(a)}{u(a)}\right],$$

and applying this formula to $u(a)$ a straightforward computation yields

$$\frac{\partial}{\partial\theta}\arg(u(a)) = \pm\frac{1}{2} + o(a),$$

where $o(a) \to 0$ when $a \to 1$. The limit equals $1/2$ when $n = 0$ and equals $-1/2$ when $n \neq 0$. Hence the total variation tends to $\pm\pi$ when the radii go to zero.

On the other hand, taking into account the asymptotic behavior (13) of f_i and h_i around $a = 1$, Figure 2 and (8), we deduce that

$$\lim_{\substack{a\to 1\\ a\in\ell_1^+}} \arg(u(a)) = \pi, \qquad \lim_{\substack{a\to 1\\ a\in\ell_2^-}} \arg(u(a)) = 2\pi, \qquad n = 0,$$

$$\lim_{\substack{a\to 1\\ a\in\ell_1^+}} \arg(u(a)) = \pi, \qquad \lim_{\substack{a\to 1\\ a\in\ell_2^-}} \arg(u(a)) = 0, \qquad n \neq 0.$$

So for any small enough circle C around $a = 1$, the set $u(C)$ meets the imaginary axis only once. Hence there is only one nodal curve approaching $a = 1$.

To finish the proof we only have to check that the nodal set of $\operatorname{Re}(u)$ does not contain the points ∞_1 and ∞_2.

A direct computation from the definition of f_i and h_i (see [13, p. 54] for details) shows that

$$\lim_{a\to\infty_i} f_j(a)a^{-1/2} = \lim_{a\to\infty_i} h_j(a)a^{1/2} = \infty, \qquad i, j = 1, 2. \tag{15}$$

Using equations (9) we obtain on ℓ_i

$$[f_1+(-1)^{i+1}f_2](a)=\mathrm{Re}(f_1)(a)=\int_{-1}^{1}\frac{x-a}{\sqrt{(x^2-1)(x-a)}}\,dx,$$
$$[h_1+(-1)^{i+1}h_2](a)=\mathrm{Re}(h_1)(a)=\int_{-1}^{1}\frac{1}{\sqrt{(x^2-1)(x-a)}}\,dx,$$

and from these equations it is easy to see

$$\begin{aligned}&\lim_{a\to\infty_i}(f_1+(-1)^{i+1}f_2)\,a^{1/2}\in\mathbb{R}_-,\\&\lim_{a\to\infty_i}(h_1+(-1)^{i+1}h_2)\,a^{1/2}\in\mathbb{R}_+,\\&\lim_{a\to\infty_i}(f_1+(-1)^{i+1}f_2)\,a^{-1/2}\\&\qquad=-\lim_{a\to\infty_i}(h_1+(-1)^{i+1}h_2)\,a^{1/2}.\end{aligned}\tag{16}$$

We consider the functions $\widetilde{f}_i(b)=f_i(1/b)$, $\widetilde{h}_i(b)=h_i(1/b)$. Then we have $\widetilde{f}_i$ and $\widetilde{h}_i$ satisfying, respectively, the differential equations

$$\widetilde{f}''(b)=-\frac{2}{b}\widetilde{f}'(b)+\frac{1}{4b^2(b^2-1)}\widetilde{f}(b),$$
$$\widetilde{h}''(b)=\frac{2b}{1-b^2}\widetilde{h}'(b)+\frac{1}{4b^2(b^2-1)}\widetilde{h}(b),$$

which are ordinary differential equations with singularities of the first kind at $b=0$. Frobenius method applied to this equations yields the fundamental solutions

$$\text{for }\widetilde{f}_i:\begin{cases}b^{-1/2}\,t_1(b),\\ b^{1/2}t_2(b)+b^{-1/2}t_3(b)\log(b),\end{cases}$$
$$\text{for }\widetilde{h}_i:\begin{cases}b^{1/2}s_1(b),\\ b^{3/2}s_2(b)+b^{1/2}s_3(b)\log(b),\end{cases}$$

where t_i, s_i are holomorphic around 0 and take real numbers into real numbers. We also have $t_1(0)$, $t_3(0)$, $s_1(0)$, $s_3(0)\neq 0$. The branch of log takes real positive numbers into real numbers.

Suppose that $a\to\infty_1$.

By equations (15), (16) and (9) we have

$$
\begin{aligned}
(\widetilde{f}_1+\widetilde{f}_2)(b) &= \lambda_1 b^{-1/2} t_1(b), && \lambda_1 \in \mathbb{R}-\{0\},\\
\widetilde{f}_1(b) &= \lambda_2 b^{1/2} t_2'(b) + \lambda_3 b^{-1/2} t_3(b)\log(b), && \lambda_3 \in i(\mathbb{R}-\{0\}),\\
(\widetilde{h}_1+\widetilde{h}_2)(b) &= \mu_1 b^{1/2} s_1(b), && \mu_1 \in \mathbb{R}-\{0\},\\
\widetilde{h}_1(b) &= \mu_2 b^{3/2} s_2'(b) + \mu_3 b^{1/2} s_3(b)\log(b), && \mu_3 \in i(\mathbb{R}-\{0\}),
\end{aligned}
$$

where t_2', s_2' are holomorphic around 0.

Suppose first that $m-n\neq 0$. Then

$$\lim_{b\to 0} u(b) = \frac{(n-m)\lambda_3 t_3(0)}{(n-m)\mu_3 s_3(0)} = \frac{\lambda_3 t_3(0)}{\mu_3 s_3(0)} \in \mathbb{R}^*.$$

If $n-m=0$, then

$$\lim_{b\to 0} u(b) = \lim_{b\to 0} \frac{b}{(1-b^2)^{1/2}} \frac{n\,(\widetilde{f}_1+\widetilde{f}_2)}{n\,(\widetilde{h}_1+\widetilde{h}_2)}(b) = -1,$$

where the last equality is a consequence of the last equation in (16).

Hence around ∞_1 there are no nodal lines of $\mathrm{Re}(u)$. Around ∞_2 we obtain the same conclusion by similar arguments. q.e.d.

Remark. The argument that the nodal lines stay away from the boundary also imply that there are no singly-periodic surfaces when $a \in (\infty,-1)\cup(1,\infty)$.

Proposition 2.3. *The nodal set of* $\mathrm{Re}(u)$ *is a connected curve in* Ω *joining* -1 *and* $+1$. *If* $nm\neq 0$, *then at least one of the points of this curve is a zero of* nf_1+mf_2. *When* $nm=0$, *there are no zeroes of* nf_1+mf_2.

Proof. We have proved in the above proposition that the nodal set C is nonvoid, stays away from the boundary of Ω minus $\{\pm 1\}$ and only one nodal curve approaches to each point $a=\pm 1$.

The set C is the union of a finite number of properly immersed curves which are embedded and smooth except at a discrete set of points where some of such curves meet in an equiangular way; see [3] for a more general setting.

Let us see first that C is connected.

Note that there are no connected components of C bounding a region Ω' such that $\overline{\Omega'}\subset\Omega$ because of the maximum principle for harmonic functions.

On the other hand, the connected components C_{-1} and C_1 at distance zero of $a = -1$ and $a = 1$ coincide. We reason by contradiction: if they are different, then consider a curve γ connecting two points in $\partial\Omega$ separating Ω in two connected components, each one containing exactly one C_i. The function $\operatorname{Re}(u)$ has a constant sign on γ. Take a curve η near $\partial\Omega$ meeting C_1 at a single point and meeting γ near its two boundary points. Then $\operatorname{Re}(u)$ has a constant sign on $\gamma \cup \eta$. But harmonic functions change sign around nodal lines. This contradiction shows that C_{-1} and C_1 coincide.

From the above two paragraphs it follows that $C_1 = C$. If there is another connected component C', then it bounds a domain Ω' such that $\overline{\Omega'} \subset \Omega$ (nodal lines can only approach the boundary at $a = \pm 1$) and we get a contradiction, which proves that C is connected.

We shall prove now that C is a regular curve, i.e., that there are no zeroes of $\nabla \operatorname{Re}(u)$ on C. In this case by elementary topological arguments it can be proved that there are at least three nodal regions of $\Omega - C$. One of these regions must be properly contained in Ω and this is not possible by the maximum principle for harmonic functions.

Let us see now that there exist zeroes of $nf_1 + mf_2$ when $nm \neq 0$.

For the case $nm \neq \pm 1$ we think of the function f_2/f_1 as taking values on the Riemann sphere $\overline{\mathbb{C}}$. By Lemma 2.1 we have $f_2(0)/f_1(0) = i$, and $\operatorname{Im}(f_2/f_1)$ restricted to the boundary of Ω is negative except at $\{1, \infty_1, \infty_2\}$ where takes the values $\{0, \pm 1\}$. As holomorphic mappings between Riemann surfaces are open, an easy connectedness argument shows that $\mathbb{R} - \{0, \pm 1\} \subset (f_2/f_1)(\Omega)$, and so for any $(n, m) \in \mathbb{Z} \times \mathbb{Z}$ such that $nm \neq 0, \pm 1$ there exists a zero of $nf_1 + mf_2$.

We consider now the case $nm = 1$. By connectedness the corresponding nodal curve meets the imaginary axis at some point a_0 which is different from ∞_i. Then $\operatorname{Re}(u(a_0)) = 0$ if and only if

$$\operatorname{Im}((f_1 + f_2)/(h_1 + h_2))(a_0) = 0,$$

but the symmetries in (10) imply that this number belongs to $i\mathbb{R}$ and so $(f_1 + f_2)(a_0) = 0$. The case $nm = -1$ can be obtained in a similar way.

In the case $nm = 0$ the nodal set is the interval $(-1, 1)$. By (8) the functions f_1 and f_2 have no zeroes on such interval. q.e.d.

Remark. The Riemann examples correspond to the case $nm = 0$; see [9].

3. Statement and proof of the main results

Using the results in the previous section we are now ready to prove

Theorem 3.1. *Let M be an embedded minimal torus with two planar ends in $\mathbb{R}^3/T$, where T is the group generated by a nontrivial translation in $\mathbb{R}^3$. Then M is one of Riemann's minimal surfaces.*

Proof. Let $M \subset \mathbb{R}^3/T$ be an embedded minimal torus with two planar ends. By the discussion in the first section we know that M can be parameterized by the Riemann surface of the polynomial $w^2 = (z^2-1)(z-a)$, with $a \in \mathbb{C} - \{\pm 1\}$ and Weierstrass data

$$g = r\,(z-a), \quad \omega = \frac{\theta}{r\,(z-a)}\frac{dz}{w}, \quad r > 0,\ |\theta| = 1.$$

Since $M \subset \mathbb{R}^3/T$, by Proposition 1.2 there exists a homology basis $\{\alpha, \beta\}$ on $\overline{M}$ such that $\mathrm{Per}_\alpha(\psi) = 0$ and $T = \langle \mathrm{Per}_\beta(\psi)\rangle$. Recall that α_1, α_2 are the lifting of the segments $[-1,a]$ and $[a,1]$, we have $\alpha = n\alpha_1 + m\alpha_2$, where m, $n \in \mathbb{Z}$, and $\gcd\{n,m\} = 1$.

If $nm \neq 0$, then by Propositions 1.2 and 2.3 there exists a continuous family of minimal surfaces joining M and a minimal surface $M_0 \subset \mathbb{R}^3/T_0$ (the one corresponding to a zero of $nf_1 + mf_2$) with vertical flux. By [14] or [20] such a surface is not embedded. An application of the maximum principle (see [14] or [20]) shows that M cannot be embedded, giving us a contradiction. This implies that $nm = 0$.

So $\alpha \in \{\alpha_1, \alpha_2\}$ and the only nodal curve of $\mathrm{Re}(u) = 0$ coincides with the interval $(-1,1)$. These surfaces are Riemann examples. q.e.d.

The following theorem is deduced from Propositions 1.1, 1.2 and 2.3.

Theorem 3.2. *Let $\overline{M}$ be a topological torus. Then for each homology basis $\{\alpha, \beta\}$ in $\overline{M}$ there exists a regular curve $\gamma_{\{\alpha,\beta\}}$, parameterized by $\mathbb{R}$, in the moduli space of conformal structures over $\overline{M}$ satisfying the following:*

(i) For each point $a \in \gamma_{\{\alpha,\beta\}}$ there exists a conformal minimal immersion ψ_a with two embedded parallel planar ends such that $\mathrm{Per}_\alpha(\psi_a) = 0$.

(ii) The family ψ_a, $a \in \gamma_{\{\alpha,\beta\}}$, is continuous.

(iii) If $\gamma_{\{\alpha,\beta\}}$ is any curve different from the one of Riemann examples, then there exists a finite number of points in $\gamma_{\{\alpha,\beta\}}$ with vertical

FIGURE 5. The fundamental domains of two surfaces in the family corresponding to $m = n = 1$ with non-vertical (left) and vertical (right) fluxes. See Figure 2 and Figure 3 for the extended surfaces.

flux. Hence there exists a deformation by minimal surfaces preserving the conformal structure; see [14].

Moreover, any singly-periodic minimal torus with two embedded parallel planar ends lies is one of the above described surfaces.

Remark. Let $\psi : M \to \mathbb{R}^3/T$ be a properly immersed minimal torus with two parallel embedded planar ends. We can assume that M is conformal to $\overline{M}_a - \{(a, 0), (\infty, \infty)\}$ and that the Weierstrass data of the immersion are given by equation (3). Suppose that $r^2|a^2 - 1| = 1$. Then the following conformal diffeomorphisms of $\overline{M}_a$

$$\sigma_1(z, w) = (z, -w),$$
$$\sigma_2(z, w) = \left(\frac{az - 1}{z - a}, \frac{(a^2 - 1)\, w}{(z - a)^2}\right),$$
$$\sigma_3(z, w) = \left(\frac{az - 1}{z - a}, -\frac{(a^2 - 1)\, w}{(z - a)^2}\right)$$

extend, respectively, to the following rigid motions:

(i) An inversion with respect to the point $\psi((1, 0))$ (or $\psi((-1, 0))$);

(ii) A reflection about a straight line orthogonal to the period, lying on a plane $x_3 =$ constant and not contained in the surface;

(iii) Either a symmetry with respect to a plane containing the period and orthogonal to the above straight line (the case $nm = \pm 1$), or a glide reflection with respect to a plane containing the period and orthogonal to the above straight line ($nm \neq 1$, including Riemann examples). In the former case, the intersection of the symmetry plane with $\psi(M)$ is a double curve on the surface and the symmetry interchange these curves.

Figure 5 shows the fundamental domains of two surfaces corresponding to the case $m = n = 1$.

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